Large-N_c and Chiral Limits of QCD and Models of the Baryon

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Two ideas have greatly contributed to our understanding of baryon structure in the framework of Quantum Chromodynamics (QCD). The first, chiral symmetry, received its fundamental justification from QCD and has been developed into the powerful technique of chiral perturbation theory. The other notion is the large-N_c limit, namely the behavior of QCD in the limit where there are a large number of colors. Like chiral symmetry, the large-N_c limit predicts many relations between baryon matrix elements. These so-called "model-independent" predictions should agree with each other, but paradoxically that is not always the case. The paradox is resolved by recognizing that the two limiting processes (large-N_c and the chiral limit) do not commute. This noncommutivity can be traced to the special role played by the Δ resonance in large-N_c QCD. This gives rise to a new parameter in the effective theory of baryons.

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1 INTRODUCTION

It has been known since the discovery of the large anomalous moment of the proton in the 1930s that the proton is not a fundamental Dirac particle; it has an internal structure. Since this time, understanding the structure of the nucleon–and of other baryons–has been an important problem in theoretical physics. Early attempts to model the structure of the baryon were based on a cloud of virtual mesons. In the 1960s an entirely different approach was developed in which the structure of the nucleon and other baryons was explained in terms of the quark model. For the past two decades, it has been understood that the nucleon is a stable state in quantum chromodynamics (QCD), the gauge theory underlying the strong interactions. Thus the structure of the nucleon is understood as arising from the interactions of quarks and gluons in QCD.

Why is the problem of baryon structure still of interest today? One primary reason is that we cannot directly solve QCD in the low energy regime and hence we cannot directly compute baryon properties from the underlying theory. Thus, at present, the theoretical study of baryons depends largely on phenomenological models and on exploiting those features of QCD which are tractable. Recently, significant progress has been made in understanding some of these tractable features of QCD including aspects of the $1/N_c$ expansion (where N_c is the number of colors) and on the chiral expansion. This colloquium is intended to review some of these developments with a particular emphasis on the implications for the various models of baryon structure that are in use.

The models discussed here are not designed to describe accurately all aspects of nucleon structure. In particular, they are meant to describe low-momentum-transfer observables such as the magnetic moment, mean square charge and magentic radii, and g_A, the axial vector current coupling constant. More generally, they should describe the low momentum form factors and transition form factors for the various currents which can couple to baryons. The models are *not* designed to predict high-momentum-transfer observables such as these form factors at large Q^2 or the structure functions which parameterize deep inelastic scattering.

It might be argued that eventually these considerations will become irrelevant. In time, numerical simulations of lattice QCD may become good enough so that reliable model-independent predictions of baryon properties may be obtained.[1] However, while high quality numerical simulations may allow us to test whether QCD can explain low-energy hadronic phenomena, they will not, by themselves, give much insight into how QCD works in the low-energy regime. Simple intuitive pictures are essential to obtain insight, and models provide such pictures. Condensed matter physics provides a useful analogy: even if one were able to solve the electron-ion many-body Schrödinger equation by brute force on a computer and directly predict observables, to have any real understanding of what is happening, one needs to understand the effective degrees of freedom which dominate the physics such as phonons, Cooper pairs, quasiparticles, and so forth. To gain intuition about these effective degrees of freedom, modeling is required. In much the same way, models of the hadrons are essential in developing intuition into how QCD functions in the low energy domain.

Of course, there are a plethora of models of the nucleon on the market. Most of these models seem, at least superficially, to be quite different. As two extreme examples consider the nonrelativistic constituent quark model (for an introduction to this model see Close (1979);

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[1]Present lattice QCD simulations of hadronic properties have significant systematic uncertainties. The present state of the art is nicely summed up in Karsch *et al.* (1995).

Isgur (1992); Karl (1992)) and the Skyrme model (Skyrme, 1961a, 1961b, 1962; there are many reviews of the Skyrme model–see for example Zahed and Brown, 1986). The degrees of freedom seem to be completely different. In one case the nucleon is thought of as a bound state of three constituent quarks, and in the other as a topological soliton comprised entirely of meson fields. (A baryon in the Skyrme model is topologically stable in much the same way that a knot in an infinite piece of string is topologically stable; one cannot "untie" the meson field configuration of a topological soliton by continuously changing the fields.)

As will be seen in the course of this colloquium, despite the apparent differences, these models share important similarities, some of which are not completely obvious. Accordingly, when assessing the models it is important to distinguish features which are peculiar to the model and those arising from general features of QCD. An obvious, trivial example of a general feature which constrains models is isospin symmetry (or, more generally, flavor symmetry). Any sensible model is isospin-symmetric to leading order and will predict a degenerate proton and neutron. The fact that a particular model predicts this degeneracy should not be viewed as a triumph of the model, but merely as a necessary condition on sensible model building.

In much the same way, the large-N_c behavior of any sensible model of baryons is strongly constrained by QCD (Witten, 1979). Moreover, this behavior, particularly the spin-flavor structure (Gervais and Sakata, 1984a, 1984b; Dashen and Manohar, 1993; Dashen $et.$ al, 1994, 1995) is highly nontrivial and one finds model-independent relations between nucleon and Δ properties which become exact as N_c gets large. (The Δ is the lowest lying non-strange baryon resonance. It has a spin of 3/2 and isospin of 3/2.) For example, the π-N-Δ coupling constant, $g_{\pi N\Delta}$ is predicted to be 3/2 times $g_{\pi NN}$, the pion-nucleon coupling constant. These relations may all be summarized in the statement: as $N_c \to \infty$, the low lying states in the system obeys a SU($2n_f$) symmetry, with n_f is the number of light flavors. This is just the symmetry of the nonrelativistic quark model with n_f quarks, each having two spin orientations. The fact that the large-N_c behavior of the model must match QCD constrains the predictions of the model up to order $1/N_c$ type corrections. In this sense, although Skyrme models and constituent quark models are quite different, both classes of models are constrained to have certain relations among observables (up to $1/N_c$ corrections) ; as far as the relations between these observables are concerned, the models are similar. Of course, in the real world $N_c = 3$ so that $1/N_c$ corrections may be large–naively one would expect they are typically of relative order 1/3. However, recently it has been observed that, for certain relationships (such as for isovector axial or magnetic couplings) these relations must hold to order $1/N_c^2$ (Dashen and Manohar, 1993; Dashen $et al.$, 1994, 1995) and thus are naively at the level of about 10%.

QCD also has approximate chiral symmetry[2] due to the fact the quark masses in the QCD Lagrangian (the "current quark masses") are much less than the typical hadronic mass scale. As is well known, apart from the explicit symmetry breaking due to the nonzero quark masses, there is also spontaneous symmetry breaking; the ground state of the theory, the vacuum, is not invariant under chiral rotations even though the Lagrangian is. This has profound consequences: Goldstone's theorem tells us that in the absence of explicit symmetry breaking there must be massless particles–particles which are identified with the pions (or more generally, with the pseudoscalar octet). Moreover, neglecting finite quark mass effects, the

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[2]Chiral symmetry is defined as the invariance of the theory under the transformation $q \to \mathrm{e}^{i\vec{\alpha}\cdot\vec{\tau}\gamma_5}q$ where q is the quark field, $\vec{tau}$ is an isospin matrix, γ_5 is a Dirac matrix and α is an arbitrary vector. This transformation mixes the upper and lower components of the Dirac spinors. The only term in the QCD Lagrangian which violates this symmetry is the quark mass term.

interactions of pions with each other and with baryons are also constrained by the symmetry to vanish at zero momentum transfer. (The constraints of chiral symmetry are nicely described in the recent review by Bernard, Kaiser and Meissner (1995).) It is quite sensible to treat the light quark masses as a perturbation and develop a systematic expansion. This expansion is chiral perturbation theory (χPT). The natural parameter is generally taken to be m_π^2/Λ^2, where m_π^2 is proportional to m_q (due to the Gell-Mann–Oakes–Renner relation (Gell-Mann *et al.*, 1968)) and Λ is a typical hadronic mass scale parameter, such as M_N, m_ρ or $4\pi f_\pi$.

Chiral symmetry strongly constrains certain properties of the nucleon and its interactions with pions, particularly π-N scattering, and one might wish to view chiral properties as providing a necessary constraint on a reasonable model of the nucleon. Many models such as Skyrme models, chiral or hybrid bag models (reviewed in Vepstas and Jackson, 1990), chiral quark meson models (reviewed in Banerjee, Broniowski and Cohen, 1987; Birse, 1991) and baryon models based on the Nambu–Jona-Lasinio model (reviewed in Alkofer, Reinhardt and Weigel, 1995) are explicitly chirally symmetric. On the other hand, many models of baryons do not respect chiral symmetry. Famous examples include both the constituent quark model and the MIT bag model (Chodos *et al.*, 1974). One attitude that can be taken is that as long as one avoids studying physics which is sensitive to low momentum π-N interactions, such as π-N scattering, the explicit role of chiral symmetry is small and its indirect effects may be simulated by a good non-chiral phenomenology.

There are certain interesting subtleties involving the large-N_c and chiral expansions. For example, in the combined large-N_c and chiral limits there are new model-independent predictions. These predictions can be derived from pion loop effects in large-N_c chiral perturbation theory–a generalization of usual chiral perturbation theory which accounts for the large-N_c behavior of the quantities involved (Cohen and Broniowski, 1992). If one looks at these predictions one immediately observes that the large-N_c and chiral limits do not always commute; for a number of quantities the qualitative result depends on whether one first takes the chiral or large-N_c limit. If such a quantity is treated as a function of m_q, then the $1/N_c$ expansion is not uniformly convergent; conversely, if such a quantity is treated as a function of N_c, then the chiral expansion is not uniformly convergent. The origin of this behavior is ultimately rather easy to trace. In the large-N_c limit the N-Δ mass splitting goes as $1/N_c$. This is seen explicitly in the Skyrme model (Adkins, Nappi and Witten, 1983) and can be derived in general from large-N_c consistency conditions (Jenkins, 1993). Consider the ratio between the pion mass and the N-Δ mass splitting:

$$d \equiv \frac{M_\Delta - M_N}{m_\pi} \ . \tag{1}$$

Clearly this ratio has the property that as one approaches the chiral limit of $m_\pi \to 0$, then d diverges; while if one approaches the large-N_c limit, then $d \to 0$. Quantities which are sensitve to the value of d can have radically different results in the two limits. It is not surprising that quantities could be highly sensitive to d. As will be discussed later, loop graphs with a virtual pion and an intermediate Δ produce such effects.

The noncommutativity of the large-N_c and chiral limits for certain observables raises some practical questions. If the two expansions are incompatible with each other for important quantities, which expansion ought one use to describe the real world? The ratio d defined in Eq. (1) is empirically approximately 2.1; *i.e.* neither large nor small. This suggests that neither the chiral nor $1/N_c$ expansions will work well for quantites which are sensitive to d.

It is far more natural to consider a combined expansion in which $1/N_c$ and m_π/Λ are both treated as small but in which $N_c m_\pi/\Lambda$ is treated as order 1. While this expansion may be the natural one, most of the models in use are not based on this expansion.

In this colloquium the relationship of chiral symmetry, large-N_c QCD, and various models of the nucleon will be discussed. The emphasis will be on models such as the Skyrme model which attempt to build in both the correct large-N_c and chiral physics. This colloquium is not intended as a comprehensive review of all facets of the problem; rather it should serve as a guide to some of the more intriguing connections between various approaches to the problem of baryon structure. The discussions here will concentrate almost entirely on the non-strange sector. The reason for this is that most of the interesting issues of principle are present for SU(2) flavor. The extension to SU(3) flavor is relatively straightforward in principle, but is less transparent–particularly when one considers the large-N_c behavior.

2   THE SKYRME MODEL, ITS HEDGEHOG RELATIVES, AND A CHIRAL PUZZLE

This section will introduce an apparent paradox concerning the Skyrme model and other models which are based on large-N_c QCD and approximate chiral symmetry. Such models include the chiral-quark meson soliton model, the chiral or hybrid bag model, and the soliton approach to the Nambu–Jona-Lasinio (NJL) model. All of these models have a few important features in common and all of the issues discussed concern these models in precisely the same way as they concern the Skyrmion.

The key features of these models are that they are all based on a mean-field theory treatment (which is ultimately justified by an appeal to large-N_c physics) and that all of these models have the long-range physics dominated by a pionic tail. It is worth stressing that all of the physics which will be discussed depends only on large-N_c physics and the role of the pion tail. Nothing in the following discussion depends explicitly on the fact that the Skyrme model is topological in nature. Thus, the various nontopological models with the baryon number carried by explicit quarks behave in the same way as the Skyrme model.

In the discussion in this section, I will restrict my attention to the traditional treatment of these models in which one works systematically in a $1/N_c$ expansion and typically only retains the leading order term. The lowest energy configuration in the mean-field theory is static and the mean-field treatment requires that the pion field be classical. As it happens, in all of these models the lowest-lying classical solutions break both the rotational and isorotational symmetry; the minium energy configurations are so-called hedgehogs with the pion's isospin direction correlated with the spatial direction (Skyrme 1961a, 1961b, 1962). Thus the pion field assumes the form

$$\vec{\pi}_a(\mathbf{r}) = \pi(r)\hat{r}_a \tag{2}$$

where the subscript a in π_a indicates the isospin component. This form is referred to as a hedgehog since the isospin points radially outward like a hedgehog protecting itself. If one rotates a hedgehog in either space or isospin space one gets a new configuration which is degenerate in energy with the original hedgehog.

Of course physical states have good angular momentum and isospin. Thus, it is clear that the hedgehog or rotated hedgehog configurations do not immediately correspond to the physical states. What then are they? To find the properties of individual states it is necessary to

project onto states of good quantum numbers. A semiclassical technique has been developed to do this which should become increasingly valid in the large-N_c limit. The technique involves the study of slowly rotating self-consistency classical configurations and then quantizing the parameters which specify the rotations (Adkins, Nappi and Witten, 1983; Cohen and Broniowski, 1986). The physical states are baryon with $I = J$. The energy splittings in the band follow a rotational spectrum:

$$E_J = E_0 + \frac{J(J+1)}{2\mathcal{I}} \tag{3}$$

Here $\mathcal{I}$, the moment of inertia, and E_0 are both of order N_c. Thus the splitting of the low-lying states in the band is of order $1/N_c^2$ relative to the total mass. The two lowest states in the band are identified as the nucleon and the delta. Predicted higher states in the band are generally assumed to be artifacts of the $1/N_c$ approximation. Moreover, the nucleon and the delta (and these exotic high spin-isospin states) all have an identical intrinsic state. Thus many observables are related (at least up to $1/N_c$ corrections).

The states in the band are labeled by the quantum numbers $I = J$, m, and m_I. Let us consider arbitrary operators, denoted by A, B, C and D which carry isospin i and angular momentum j, with third components μ_i and μ. The following relations among matrix elements are true:

$$
\begin{aligned}
\langle I, m, m_I | A^{i=0,j=0} | I, m, m_I \rangle &= a \\
\langle I, m, m_I | B^{i=1,j=0}_{\mu_i=0} | I, m, m_I \rangle &= b \, m_I \\
\langle I, m, m_I | C^{i=0,j=1}_{\mu=0} | I, m, m_I \rangle &= c \, m \\
\langle I', m', m'_I | D^{i=1,j=1}_{\mu,\mu_I} | I, m, m_I, \rangle &= d \, X^{\mu'\mu}_{(I',m'm'_I)(I,m,m_I)}
\end{aligned}
\tag{4}
$$

where the lower case coefficients a, b, c, d are constants which do not depend on the particular state $|I, m, m_I\rangle$ in the band, and

$$X^{\mu'\mu}_{(I',m'm'_I)(I,m,m_I)} \equiv \sqrt{\frac{2I'+1}{2I+1}} \begin{pmatrix} I & 1 & I' \\ m_I & \mu_i & m'_I \end{pmatrix} \begin{pmatrix} I & 1 & I' \\ m & \mu & m' \end{pmatrix} \tag{5}$$

These relationships were recognized much earlier to be "model independent" in the sense that all Skyrme models, and other large-N_c hedgehog models will satisfy these relations. In fact, these relations can be tested for a number of observables and they appear to work quite well (Adkins and Nappi, 1985). However, these results are model-independent in a much deeper sense–consistency of the large-N_c expansion requires them to be true (Gervais and Sakita, 1984a, 1984b; Dashen and Manohar 1993; Jenkins, 1993; Dashen, Jenkins, and Manohar 1994, 1995; Broniowski, 1994). This large-N_c consistency argument is outlined in Appendix B.

Of course, while ratios between various observables can be fixed from Eqs. (4), the actual values of the coefficients on the right side of Eqs. (4) depend in detail on the dynamics of the model. In general, for arbitrary operators with real world parameters one must simply calculate and see how well the models do. However, as was noted previously, the models are designed with chiral symmetry in mind. The models have a well-defined chiral limit corresponding to vanishing quark mass, or equivently m_π^2 vanishing. It is interesting to see how predictions of the model behave as one approaches this limit.

The pion is the lightest degree of freedom in the problem. Thus, the classical pion field of Eq. (2) at long distance behaves like a p-wave Yukawa field with the radial dependence:

$$\vec{\pi}_a(\mathbf{r}) \to \vec{\pi}_a^{\text{asympt}}(\mathbf{r}) = \hat{\mathbf{r}}_a \frac{3g_{\pi NN}}{8\pi M_N}(m_\pi + 1/r)\frac{e^{-m_\pi r}}{r}. \tag{6}$$

Here $g_{\pi NN}$ is the π-N coupling constant as determined in the model, and the factor of $3/8$ emerges from the details of the semiclassical projection. Some observables depend strongly on the tail of the configuration. If the tail is weighted strongly enough for some quantity then its predicted value will either diverge as $m_\pi \to 0$ or go as m_π. In such cases, one can obtain essentially model-independent predictions for the behavior of the observables as one approaches the chiral limit. A few of these predictions are listed in Table 1. The derivation of these relations is semi-classical in nature; the derivation of one of these quantities is outlined in Appendix A. These predictions depend only on m_π, and $g_{\pi NN}/M_N$; the vector-isovector observables also depend on the Δ nucleon mass splitting. All of these predictions are "model independent" in the sense mentioned above: they apply to any large-N_c hedgehog model without regard to the details of the dynamics. There is implicit model dependence in the actual predictions since the relations depend on $g_{\pi NN}/M_N$ (and perhaps on $M_\Delta - M_N$) as given by the models, and the models are not guaranteed to produce the correct value for $g_{\pi NN}/M_N$ or $M_\Delta - M_N$. Moreover, these model-independent predictions only apply to the leading chiral nonanalytic part of the expressions for observables. In addition, there are chirally suppressed contributions which depend explicitly on the details of the model; for realistic values of m_π these chirally suppressed contributions may be numerically significant for some observables.

Now we come to the puzzle: One can evaluate all of these quantities using chiral perturbation theory (χPT). The approach is reviewed in Bernard, Kaiser and Meissner (1995). The leading chirally singular piece is determined entirely from pion one loop graphs and the coefficients are fixed at lowest order in the theory. The strong belief is that chiral perturbation theory should exactly reproduce QCD in the limit as $m_\pi^2 \to 0$. Thus, the χPT calculation of the various chirally singular properties in Table 1 should be exact. However, in looking at Table 1, one sees that all of these χPT predictions disagree with the predictions of the hedgehog models.

Obviously something is wrong. Two apparently model-independent calculations of the same quantity give different results. One possibility, of course, is that one approach or the other is simply wrong. A more interesting possibility is that each of these predictions is in fact correct in its domain of validity, but that the domains of validity are mutually inconsistent. Before exploring this possibility in detail it is worth noting that there is a hint that something strange is going on. If one looks at the ratio of the hedgehog model predictions of the leading chiral behavior to the conventional χPT behavior one sees that this ratio is identical for all quantities with the same quantum numbers. For any chirally divergent quantity A with angular momentum and isospin quantum numbers of J and I in Table 1.

$$\frac{A_{\text{hedgehog}}^{I=J=0}}{A_{\chi\text{PT}}^{I=J=0}} = 3 \tag{7}$$

$$\frac{A_{\text{hedgehog}}^{I=1,J=1}}{A_{\chi\text{PT}}^{I=1,J=1}} = \frac{3}{2} \tag{8}$$

This universal, but quantum number dependent, behavior suggests quite strongly that there is some deep connection between the hedgehog models and χPT despite the fact that they do not agree.

How can it be that the domains of validity of the two approaches are mutually incompatible? The hedgehog models were based on working to leading order in the large-N_c approximation and then subsequently taking the chiral limit. In contrast, the χPT predictions are based solely on the chiral limit. Thus, the possibility exists that for these chiral singular quantities, the large-N_c limit and the chiral limit do not commute. Indeed this possibility was recognized more than ten years ago by Adkins and Nappi (1983), who conjectured that this noncommutativity explained the discrepancy between the Skyrme model and χPT predictions for the isovector charge radius. An interesting development in recent years is that this conjecture has been shown to be correct and, more importantly, the dynamics underlying the noncommutativity has been shown to arise from the behavior of the Δ resonance (Cohen and Broniowski, 1992; Dashen, Jenkins and Manohar, 1994; Cohen, 1995).

3   BARYONS IN THE LARGE N_c AND CHIRAL LIMITS

To see why the Δ plays such a pivotal role for these quantities in the large-N_c limit, it is useful to go back to χPT and ask why it is that one pion loop graphs give the leading chiral singularity for a generic quantity. The quantities being studied can be thought of as the response of the system when it is perturbed by some external probe. The response can be calculated via perturbation theory. The chiral singularities come from the fact that in this perturbation calculation, there are small energy denominators associated with nucleon-plus-one-pion states. As $m_\pi \to 0$, the π-N states with arbitrarily low relative momentum give energy denominators which are arbitrarily small. If the phase space for such states is large enough these small energy denominators can cause divergences in the chiral limit of $m_\pi = 0$. Thus, the chiral singularities typically arise from Feynman graphs such as diagram a of Fig. (1).

The implicit assumption in derving the χ PT expressions in Table 1 is that the only states which are arbitrarily low in energy are the nucleon-plus-pion states. However, as one approaches the large-N_c limit, this ceases to be true. As N_c becomes large, the Δ becomes nearly degenerate with the nucleon. In the hedgehog models Eq. (3) gives $M_\Delta - M_N = 3/\mathcal{I}$ where $\mathcal{I} \sim N_c$ so that $M_\Delta - M_N \sim 1/N_c$. Moreover, the fact that $M_\Delta - M_N \sim 1/N_c$ is model independent; it can be derived using the large-N_c consistency relations (Jenkins, 1993). Thus, the assumption that nucleon plus pion states are the only low-lying excitations of the nucleon is wrong as one approaches the large-N_c limit.

In the context of the large-N_c limit, we can now revisit the argument that the one pion loop contributions dominate the chiral singular quantities. It is essentially the same argument before; however, in addition to graphs such as those in diagram a of Fig. 1 where there is a π-N intermediate state, there are also π-Δ intermediate states (diagram b). Since the N and Δ are degenerate in the large N_c limit, the loops containing a Δ also have anomalously small energy denominators which can yield results which diverge as $m_\pi \to 0$. Moreover in the large-N_c limit, the π-N-Δ coupling is completely determined from the π-N-N coupling (or equivalently from g_A and various Clebsch-Gordan factors). For large N_c, both the nucleon

and Δ are arbitrarily heavy and can be treated as static. Since the pion couples to a static source derivatively, the vertex is an operator with $I = 1$ and $J = 1$; the relative strengths of the pion's coupling to various baryons is given by the fourth equation of eqs. (4).

Now consider the contributions to some operator from diagrams a and b of Fig. 1 in the large-N_c limit. The energy denominators in the two diagrams become identical as $N_c \to \infty$ since $(M_\Delta - M_N) \to 0$. The only difference between diagrams a and b is an overall factor coming from the vertices. The net contributions of these depend on the quantum numbers of the observables since these determine how the various Clebsch-Gordan factors in the vertices are combined. Explicitly doing these sums one finds that the contributions for an operator $A^{I,J}$ (with isospin and angular momentum quantum numbers I and J) from the π-N loops (as in diagram a. of Fig. 1) and π-Δ loops (as in diagram b.) are related as follows:

$$
\begin{aligned}
A_{\pi-\Delta}^{I=J=0} &= 2\, A_{\pi-N}^{I=J=0} \\
A_{\pi-\Delta}^{I=J=1} &= \frac{1}{2}\, A_{\pi-N}^{I=J=1} \\
A_{\pi-\Delta}^{I=1,J=0} &= -1\, A_{\pi-N}^{I=1,J=0}
\end{aligned}
\tag{9}
$$

The large-N_c limit of chirally singular properties includes both of these contributions. Adding them together and noting that the π-N contribution is the conventional χPT prediction for the chirally singular part, one finds

$$
\begin{aligned}
A_{\text{Large}N_c}^{I=J=0} &= 3\, A_{\chi\text{PT}}^{I=J=0} \\
A_{\text{Large}N_c}^{I=J=1} &= \frac{3}{2}\, A_{\chi\text{PT}}^{I=J=1}
\end{aligned}
\tag{10}
$$

Comparing Eqs. (10) with Eqs. (8) we see immediately that the large-N_c version of chiral perturbation theory predicts precisely what is calculated in the hedgehog models.

For $I = 1, J = 1$ operators one finds that contributions from diagram b cancel those from diagram a. Thus, for example, the $\ln(m_\pi)$ term in the isovector charge radius should vanish owing to such a cancellation (Cohen, 1995). From Table 1 we see that the $\ln(m_\pi)$ term is not present in the hedgehog model predictions.

The Skyrme models reproduce the chirally singular behavior of chiral perturbation theory, provided one uses the large-N_c generalization of chiral perturbation theory. This is deeply satisfying from a purely theoretical perspective. At a practical level, however, there is still a problem with these models. The preceding analysis was done in the large-N_c limit, which implicitly took the N-Δ mass splitting to be small compared to all energies of order N_c^0. Unfortunately, m_π is of order N_c^0 and in the real world, it is not much larger than $M_\Delta - M_N$. Indeed it is a factor of two smaller. The preceding analysis is formally correct but it is of questionable utility.

The preceeding analysis can be generalized to take into account the fact that the $M_\Delta - M_N \sim m_\pi$. Formally, this corresponds to considering an expansion in which $1/N_c$ and m_π/Λ are each taken as small, with $N_c m_\pi$ treated as $\mathcal{O}(1)$. Analysis based on this generalized expansion is large N_c chiral perturbation theory. In practice, this expansion implies that the leading order contribution for chirally singular quantities again comes from the sum of diagrams a and b in Fig. (1). However, the Δ mass is not taken to be degenerate with the nucleon mass. Rather, the mass difference is kept to all orders. The diagrams can be evaluated

without difficulty; the results are expressed most simply in terms of a universal function $S(d)$ where $d \equiv (M_\Delta - M_N)/m_\pi$ and $s(d)$ is given by

$$
\begin{aligned}
s(d) &\equiv \frac{4}{\pi} \frac{1}{\sqrt{1-d^2}} \tan^{-1}\left(\sqrt{\frac{1-d}{1+d}}\right) \quad \text{for } d < 1 \\
s(d) &\equiv \frac{4}{\pi} \frac{1}{\sqrt{d^2-1}} \tanh^{-1}\left(\sqrt{\frac{d-1}{d+1}}\right) \quad \text{for } d > 1
\end{aligned}
\tag{11}
$$

The leading order results for a number of quantities are given in Table 2.

The function $s(d)$ has some interesting properties. As $d \to 0$ (*i.e.* large-N_c limit), $s(d) \to 1$. If one replaces $s(d)$ by unity for any of the quantities in Table 2 one reproduces the analogous Skyrme model results. As $d \to \infty$ (*i.e* the chiral limit), $s(d)$ goes to zero as $2\ln(d)(\pi d)^{-1}$. As the chiral limit is approached, the Δ loop contributions become small compared to the nucleon loop and one reproduces the naive χPT results. Thus, the large-N_c χPT calculations smoothly interpolate between the naive large-N_c results of the hedgehog models and the results of conventional χPT.

In the real world, $d \approx 2.1$ and $s(d) \approx .47$. Thus $s(d)$ is just about half way between the large-N_c value of unity and the chiral value of zero. To the extent one studies properties for which $s(d)$ plays an important role (which includes all leading chirally nonanalytic properties), low order expansions around either the chiral limit or the large-N_c limit will be incorrect. We are far from both.

4 IMPLICATIONS FOR MODELING BARYONS

What does the preceding analysis tell us about modeling baryons? Clearly one should use great care when interpreting the results of the standard Skyrme model (or other hedgehog model) calculations for quantities in which the long-distance part of the pion tail is important. Errors due to the large-N_c limit (upon which calculations in the model are based) are likely to be very large since in these calculations $s(d)$ is implicitly taken to be unity. What alternatives are there?

One possibility is simply to declare the long-range chiral physics uninteresting and beyond the scope of the model. The hope is that by suitable choice of phenomenological parameters one can simulate the correct long-distance physics without including the explicit pionic degrees of freedom. This is the strategy underlying the constituent quark model. On the other hand, the long distance chiral physics is one of the few nonpertubative aspects of QCD that we actually understand. If we hope to learn something fundamental from the models about how QCD plays out in the low energy regime, we should at least be able to explain these simple features. In this context, it is also worth remarking that the relationship of the constituent quark model to QCD remains quite cloudy.

Another alternative is to simply abandon the enterprise of modeling entirely. One can develop large-N_c chiral perturbation theory in a systematic and model-independent way. At any given order in the expansion, one will have only a finite number of parameters, which should be fitted experimentally. Having fit these parameters, one can then proceed to predict other quantities. Initial steps in this program have been taken ,and the results are promising (see, for example, Luty and March-Russell (1995); Jenkins and Lebed (1995); Bedaque and

Luty (1995); Jenkins (1995)). However, this approach has at least one important limitation: the regime of validity of the expansion is unknown. It is not known how high in momentum transfer the theory will give useful results.

A final approach would be to consider models which formally give both the large-N_c and chiral limits correctly, and which does not implicitly assume that0 $M_\Delta - M_N$ is either much smaller or much larger than m_π. One class of models which does this includes the cloudy bag model (for a review see Thomas (1983)), which has a quark core to which one adds explicit quantum pion loops. Because the pions are treated quantum mechanically from the outset, they automatically get the full function $s(d)$ as $N_c \to \infty$ and $m_\pi \to 0$. One can enhance the constituent quark model in a similar way by including explicit pion loops.

Another possibility is to develop a new calculational scheme for studying the Skyrme model and its hedgehog cousins. The analysis of the Skyrme model, which showed inconsistencies with the chiral behavior, was based on the conventional calculation scheme, which implies a $1/N_c$ expansion from the outset. If one could consistently reformulate the calculation of quantities in the Skyrme model in a fashion consistent with the expansion scheme of large N_c chiral perturbation theory there would be no difficulty. Is there a calculational scheme consistent with an expansion? The answer, apparently, is that there is. This scheme was recently introduced by Dorey, Hughes and Mattis (1994). It has been christened the "rotationally improved skyrmion" (RISKY) and is based on a functional integral formulation around a hedgehog distorted by its rotational motion. One test of the consistency of this approach is its ability to reproduce the physics associated with the ratio of the Δ nucleon mass splitting to m_π. Using RISKY techniques, the functional dependence on this ratio, i.e., $s(d)$, was reproduced for one of the quantities in Table 2 (Dorey, Hughes and Mattis, 1995), and presumably the appropriate $s(d)$ factors for other quantities can also be obtained. A major drawback of the RISKY approach is that it is quite cumbersome. Indeed, at present, the RISKY approach has not yet been implemented fully for any particular Skyrme model Lagrangian in the sense that the various low energy observables have not been computed in terms of the parameters in the Lagrangian.

The general question of how to model the low energy properties of baryons remains an interesting and vibrant field of study. The connection of these models to QCD is clearly a central issue. In this context, it is important that good models should, at a minimum, reproduce those aspects of QCD which we actually understand. Fortunately, in the past several years our understanding of certain nonperturbative aspects of QCD has increased; particularly the large-N_c and chiral properties of baryons. Serious models should build upon this understanding.

5   ACKNOWLEDGMENTS

A   CHIRAL PROPERTIES IN SKYRME MODELS

In this appendix, it is shown how the "model-independent" predictions of chirally nonanalytic quantities are calculated in Skyrme and other large-N_c hedgehog models. The basic strategy is to use the standard semiclassical treatment (Adkins, Nappi, and Witten 1983; Cohen and Broniowski, 1986). In this approach, the various operators are treated as classical with the classical fields from the hedgehog configuration. The only quantum mechanics come from the quantization of the collective rotational (or, equivalently, isorotational) modes. The important thing to note is that if a quantity diverges in the chiral limit, it implies that the quantity is dominated by the long-range pionic tail, whose range goes to infinity as $m_\pi \to 0$. Thus for the chirally divergent part of these quantities, it is sufficient to pick up the contributions due to the long-range part of the pions.

Consider as a concrete example the isovector charge radius. The pion fields in an arbitrary hedgehog configuration may be written as $\pi_a(\vec{x}) = \pi(r)\hat{r}_a$, where a is the isospin direction. In any of these hedgehog models the pion field tends asymptotically to a p-wave Yukawa form with a field strength fixed by $g_{\pi NN}$ as given by eq. (6) where $g_{\pi NN}$ is the model's π-N coupling. Following Cohen and Broniowski (1986), the asymptotic pion field contribution to the isovector charge radius is given by

$$\langle r^2 \rangle_{I=1} \;=\; \frac{1}{\mathcal{I}} \int \mathrm{d}r \, r^4 \frac{8\pi}{3}((\pi_\mathrm{asympt})^2 \;\; , \tag{12}$$

where $\mathcal{I}$ is the moment of inertia. Other contributions to $\langle r^2 \rangle_{I=1}$ come from shorter-ranged effects, which are model-dependent and do not diverge in the chiral limit. The expression in Eq. (12) is easy to understand–it is the asymptotic pion field contribution to the semiclassical isovector-vector current multiplied by r^2. Evaluating the integral and using the relation of Goldberger and Treiman (1958), $g_{\pi NN} f_\pi = g_A M_N$ (which is true to leading order in the pion mass) and using the fact that for hedgehog models $1/\mathcal{I} = 2/3(M_\Delta - M_N)$ (to leading order in the $1/N_c$ expansion), immediately yields

$$\langle \mathrm{N}|r^2|\mathrm{N} \rangle_{I=1} \;=\; \frac{5\,g_A^2\,\delta m}{16\,\pi\,f_\pi^2\,m_\pi} \tag{13}$$

plus corrections which are higher order in either $1/N_c$ or m_q.

Model-independent relations for other quantities in Table 1 can be derived in an analogous fashion.

B   CONSISTENCY AND LARGE N_c

This appendix briefly discusses the physical basis for the large-N_c consistency relations originally derived by Gervais and Sakita (1984a,1984b) and subsequently rediscovered and extended by Dashen and Manohar (1994). There has been considerable work on this subject in the past several years (Jenkins 1993; Dashen, Jenkins and Manohar, 1994,1995; Broniowski, 1994). In this appendix, mathematical detail will be omitted for reasons of space.

The basic argument is quite simple. Consider π-N scattering. From Witten's large-N_c analysis of generic baryons (Witten, 1979), we expect that the effective π-N coupling for derivatively coupled pions, $g_{\pi NN}/M_N = g_A/f_\pi$, is of order N_c. This implies that the Born and

crossed-Born contributions in Fig. 1 are of order N_c^2. Moreover, the non-Born terms have a different energy dependence owing to the presence of a mass difference in the propagator and cannot cancel out the Born and crossed-Born contributions. This presents a serious problem: as $N_c \to \infty$, the Born and crossed-Born contributions become arbitrarily large and, unless they cancel each other, violate unitarity.

It is easy to see that these contributions might cancel. In the large N_c limit, the nucleon is very massive and does not recoil. Thus, the energy denominator in the Born and crossed-Born terms are equal and opposite. Indeed, for scalar or isoscalar mesons, this fact alone is sufficient to guarantee cancelation. However, the pion coupling is a p-wave isovector and thus, in the nonrelativistic limit–which is appropriate at large N_c–goes as $N^\dagger \sigma_i \tau_a N \partial_i \pi^a$. Thus the sum of the Born and crossed-Born terms is proportional to $N_c^2 [\sigma_i \tau_a, \sigma_j \tau_b]$, which is nonzero. How then can one have consistency between unitarity and Witten's large-N_c prediction for the scaling of the π-N coupling constant?

The answer is straightforward. If the Δ were degenerate with the nucleon in the large-N_c limit, then the Δ graphs could cancel the nucleon Born graphs. Taking into consideration π-Δ scattering, one deduces the need for an additional $I = J = 5/2$ state to keep the amplitude unitary. The existence of the entire tower of states that is found in the Skyrme model is deduced. If one uses a vector-isovector operator, $X_{i,a}$, to describe all of the π-B-B' couplings, where B and B' are baryons in the tower, then the consistency condition implies

$$[X_{i,a}, X_{j,b}] = 0 \ \ . \tag{14}$$

From this equation, a simple recursion relation follows and allows the computation of matrix elements of X in terms of Clebsch-Gordan coefficients.

To obtain other consistency relations one can apply similar reasoning to scattering amplitudes for $\pi + B \to \pi + m + B$, where m is a meson with fixed quantum numbers. In the special case where m is a pion, one finds that model-independent relations hold to relative order $1/N_c^2$. One can apply the same technique to the reaction $\pi + B \to \pi + \gamma + B$, which constrains electromagnetic coupling, and to processes which involve weak interactions.

REFERENCES

Adkins, G. S. , C. R. Nappi, 1984, Nucl. Phys. **B 233**, 109.

Adkins, G. S. , C. R. Nappi, 1985, Nucl. Phys. **B 249**, 507.

Adkins, G. S. , C. R. Nappi, and E. Witten, 1983, Nucl. Phys. **B 228**, 552.

Alkofer, R., H. Reinhardt and H. Weigel, 1995, "Baryons as Chiral solitons in the Nambu–Jona-Lasinio model", to appear in Phys. Rep. **263**; hep-ph/9501213.

Banerjee, M. K., W. Broniowski, and T. D. Cohen, 1987, in *Chiral Solitons*, edited by K.-F. Liu (World Scientific, Singapore), p. 255.

Bernard, V., N. Kaiser and U.-G. Meissner, 1995, Int. J. Mod. Phys. **E 4**, 193.

Birse, M. C., 1991, Prog. in Part. and Nucl. Physics **25**, 1.

Bedaque, P. F., and M. Luty, 1995, "Baryon masses at second order in large-N chiral perturbation theory", hep-ph/9510453.

Broniowski, W., 1994, Nucl. Phys. **A 580**, 429.

Chodos, A., R. L. Jaffe, K. Johnson, C. B. Thorn, V.F. Weisskopf, 1974, Phys. Rev. **D 9** 3471.

Close, F., 1979, *An Introduction To Quarks and Partons*, (Academic Press, London).

Cohen, T. D, 1995, Phys. Lett. **B 359**, 23.

Cohen, T. D. and W. Broniowski, 1986, Phys. Rev. **D 34**, 3472.

Cohen, T. D. and W. Broniowski, 1992, Phys. Lett. **B 292**, 5.

Dashen, R., E. Jenkins and A. V. Manohar, 1994, Phys. Rev. **D 49**, 4713.

Dashen R., E. Jenkins and A. V. Manohar, 1995, Phys. Rev. **D 51**, 2489.

Dashen R. and A. V. Manohar, 1993, Phys. Lett.**B 315**, 42.

Dorey, N., J. Hughes, M. P. Mattis, 1994, Phys. Rev. **D 50**, 5816.

Dorey, N., J. Hughes, M. P. Mattis and D. Skidmore, 1995, "Chiral divergent properties of hadrons in the large-N_c limit" Michigan State Preprint MSU-50605; hep-ph/9509310.

Gell-Mann, M., R. J. Oakes and B. Renner, 1968, Phys. Rev. **175**, 2195.

Gervais, J. L and B. Sakita, 1984a, Phys. Rev. Lett. **52**, 87.

Gervais, J. L and B. Sakita, 1984b, Phys. Rev. **D 30**, 1795.

Goldberger, M. and S. Treiman, 1958, Phys. Rev. **111**, 354.

Isgur, N., 1992, in *Nucleon Resonances and Nucleon Structure*, editied by G. A. Miller (World Scientific, Singapore), p. 45.

Jenkins, E., 1993, Phys. Lett. **B 315**, 441.

Jenkins, E. and R. F. Lebed, 1995, Phys. Rev. **D 52**, 282. Jenkins, E., 1995, "Chiral Lagrangian for Baryons in the $1/N_c$ Expansion" , University of California San Diego preprint UCSD/PTH 95-17; hep-ph/9509433

Karl, G., 1992, in *Nucleon Resonances and Nucleon Structure*, edited by G. A. Miller (World Scientific, Singapore), p. 71.

Karsch, F., J. Engels, E. Laermann, B. Petersson, 1995, Eds. *Lattice '94. Proceedings , 12th International Symposium On Lattice Field Theory* (North-Holland, Amsterdam, Netherlands) (Nucl. Phys. B, Proc. Suppl. 42).

Luty, M. A. and J. March-Russell, 1995, Nucl. Phys, **B 426**, 2322.

Skyrme, T. H. R., 1961a, Proc. R. Soc. London **260**, 126.

Skyrme, T. H. R., 1961b, Proc. R. Soc. London **261**, 237.

Skyrme, T. H. R., 1962, Nuc. Phys, **31**, 556.

Thomas, A. W., 1983, *Advances in Nuclear Physics* **13**, edited by J. Negele and E. Vogt (Plenum, New York), p. 1.

Witten, E., 1979, Nuc. Phys **B 160**, 57.

Vepstas, L. and A. D. Jackson, 1990, Phys. Rep. **187**, 109.

Zahed, I and G. E. Brown, 1986, Phys. Rept. **142**, 1.

Table 1: The leading chiral nonanalytic contribution for a number of nucleon properties. These contributions will dominate the full answer in the limit $m_\pi \to 0$. The column headed "hedgehog models" gives the "model-independent" predictions for the Skyrme model or any other large-N_c hedgehog model. The column headed "Naive χPT" gives the prediction from chiral perturbation theory based on the assumption that the pion is the only light excitation in the problem. All of these quantities are experimentally observable except for those labeled with a †. These quantities concern the quark mass dependence of observables; for convenience these quantities are re-expressed in terms of pion mass dependences. In principle these quantities could be extracted from numerical simulations of QCD.

Quantity	Quantum#'s	Hedgehog Models	Naive χPT
scalar radius $\langle r^2 \rangle_s = 6 \frac{\partial\, \sigma(t)}{\partial\, t}\big|_{t=0}$	$I=0$ $J=0$	$\frac{g_a^2}{f_\pi^2} \frac{9 m_\pi}{64\pi}$	$\frac{g_a^2}{f_\pi^2} \frac{3 m_\pi}{64\pi}$
quark mass dependence of M_N (in terms of m_π^2) $\frac{\partial^2 M_N}{\partial (m_\pi^2)^2}$ †	$I=0$ $J=0$	$-\frac{g_a^2}{f_\pi^2} \frac{27}{128\pi m_\pi}$	$-\frac{g_a^2}{f_\pi^2} \frac{27}{128\pi m_\pi}$
electric polarizability of nucleon α_N	$I=0$ $J=0$	$\frac{g_a^2}{f_\pi^2} \frac{5 e^2}{128\pi^2 m_\pi}$	$\frac{g_a^2}{f_\pi^2} \frac{5 e^2}{384\pi^2 m_\pi}$
magnetic polarizability of the nucleon β_N	$I=0$ $J=0$	$\frac{g_a^2}{f_\pi^2} \frac{e^2}{256\pi^2 m_\pi}$	$\frac{g_a^2}{f_\pi^2} \frac{e^2}{768\pi^2 m_\pi}$
isovector magnet radius of nucleon $\langle r^2 \rangle^{I=1}_{\text{mag}} = \frac{6}{\kappa_{I=1}} \frac{\partial F_2(t)}{\partial t}\big|_{t=0}$	$I=1$ $I=1$	$\frac{1}{\mu^{\text{anom}}_{I=1}} \frac{g_a^2}{f_\pi^2} \frac{3}{8\pi m_\pi}$	$\mu^{\text{anom}}_{I=1} \frac{g_a^2}{f_\pi^2} \frac{1}{4\pi m_\pi}$
quark mass dependence of anomalous isovector moment (in terms of m_π^2) $\frac{\partial \mu^{\text{anom}}_{I=1}}{\partial m_\pi^2}$ †	$I=1$ $J=1$	$\frac{g_a^2}{f_\pi^2} \frac{3}{8\pi m_\pi}$	$\frac{g_a^2}{f_\pi^2} \frac{1}{4\pi m_\pi}$
nucleon isovector chargeradius $\langle r^2 \rangle^{I=1}_{\text{elec}} = 6 \frac{\partial F_1(t)}{\partial t}\big|_{t=0}$	$I=1$ $J=0$	$\frac{g_a^2}{f_\pi^2} \frac{M_\Delta - M_N}{m_\pi} \frac{5}{16\pi}$	$\frac{5 g_a^2 + 1}{f_\pi^2} \frac{\ln(m_\pi)}{8\pi^2}$

Table 2: The leading chiral nonanalytic contribution for a number of nucleon properties in large-N_c chiral perturbation theory. These contributions will dominate the full answer in the limit $m_\pi \to 0$, $N_c \to \infty$, $N_c m_\pi \to$ constant. The function $s(d)$ is given in Eq. (11); $d \equiv (M_\Delta - M_N)/m_\pi$. All of these quantities are experimentally observable directly except for those labeled with a $\dagger$.

Quantity	Quantum#'s	Large $N_c\chi$PT
scalar radius $\langle r^2 \rangle_s = 6\frac{\partial\sigma(t)}{\partial t}\vert_{t=0}$	$I=0$ $J=0$	$(1 + 2\,s(d)\,)\frac{g_a^2}{f_\pi^2}\frac{3m_\pi}{64\pi}$
quark mass dependence of M_N (in terms of m_π^2) $\frac{\partial^2 M_N}{\partial(m_\pi^2)^2}$ $\dagger$	$I=0$ $J=0$	$-(1 + 2\,s(d)\,)\frac{g_a^2}{f_\pi^2}\frac{27}{128\pi m_\pi}$
electric polarizability of nucleon α_N	$I=0$ $J=0$	$(1 + 2\,s(d)\,)\frac{g_a^2}{f_\pi^2}\frac{5e^2}{384\pi^2 m_\pi}$
magnetic polarizability of the nucleon β_N	$I=0$ $J=0$	$(1 + 2\,s(d)\,)\frac{g_a^2}{f_\pi^2}\frac{e^2}{768\pi^2 m_\pi}$
isovector magnet radius of nucleon $\langle r^2 \rangle_{\mathrm{mag}}^{I=1} = \frac{6}{\kappa_{I=1}}\frac{\partial F_2(t)}{\partial t}\vert_{t=0}$	$I=1$ $I=1$	$(1 + \frac{1}{2}s(d)\,)\mu_{I=1}^{\mathrm{anom}}\frac{g_a^2}{f_\pi^2}\frac{1}{4\pi m_\pi}$
quark mass dependence of anomalous isovector moment (in terms of m_π^2) $\frac{\partial\mu_{I=1}^{\mathrm{anom}}}{\partial m_\pi^2}$ $\dagger$	$I=1$ $J=1$	$(1 + \frac{1}{2}s(d)\,)\frac{g_a^2}{f_\pi^2}\frac{1}{4\pi m_\pi}$
nucleon isovector charge radius $\langle r^2 \rangle_{\mathrm{elec}}^{I=1} = 6\frac{\partial F_1(t)}{\partial t}\vert_{t=0}$	$I=1$ $J=0$	$d\,s(d)\frac{g_a^2}{f_\pi^2}\frac{5}{16\pi}$

Figure 1: Typical pion loop diagrams which contribute to the leading chiral nonanalytic behavior for nucleon properties in the large-N_c limit of QCD. The cross represents the particular external source being probed–electric, magnetic, scalar, *etc.* Diagram a. gives the leading nonanalytic behavior as one approaches the pure chiral limit ($m_\pi \ll (M_\Delta - M_N)$). Diagram b. becomes comparable when $m_\pi \approx (M_\Delta - M_N)$.

Figure 2: Born and crossed-Born contributions to π-N scattering.

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